A Brief History of Numerical Algebraic Geometry

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Surveys of Numerical Algebraic Geometry

- Reference on the area up to 2005:
  - A.J. Sommese and C.W. Wampler, ***Numerical solution of systems of polynomials arising in engineering and science***, (2005), World Scientific Press.
- Survey up to 2010 oriented towards Kinematics
  - C.W. Wampler and A.J. Sommese, ***Numerical Algebraic Geometry and Algebraic Kinematics***, Acta Numerica 20 (2011), 469-567.
- Up to 2013 oriented towards Bertini
  - D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, ***Numerically solving polynomial systems with Bertini***, (2013), SIAM.
- Developments related to systems of PDEs upto 2013
  - W. Hao, B. Hu, and A.J. Sommese, ***Numerical algebraic geometry and differential equations***, in Future Vision and Trends on Shapes, Geometry and Algebra, ed. by R. De Amicis and G. Conti, Springer Proc. in Mathematics & Statistics, Vol. 84 (2014), 39-54.

Numerical algebraic geometry grew out of

- Continuation methods for computing isolated solutions of polynomial systems
- Classical methods to studying a positive dimensional algebraic sets by studying the slices of the set.

Computing Isolated Solutions of Polynomials Systems

Find all solutions of a polynomial system on C^N:

$$\begin{bmatrix} \mathrm{f}_1(x_1,\ldots,x_N) \\ \vdots \\ \mathrm{f}_N(x_1,\ldots,x_N) \end{bmatrix} = 0$$

Why?

To solve problems from engineering and science.

Characteristics of Engineering Systems

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- systems depend on parameters: typically they need to be solved many times for different values of the parameters.
- usually only real solutions are interesting.
- usually only finite solutions are interesting.
- nonsingular isolated solutions *were* the center of attention.

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Homotopy continuation is our main tool:

Start with known solutions of a known start system and then track those solutions as we deform the start system into the system that we wish to solve.

Path Tracking

This method takes a system g(x) = 0, whose solutions we know, and makes use of a homotopy, e.g.,

$$H(x, t) = (1 - t)f(x) + tg(x).$$

Hopefully, H(x,t) defines "paths" x(t) as t runs from 1 to 0. They start at known solutions of g(x) = 0 and end at the solutions of f(x) at t = 0.

The paths satisfy the Davidenko equation

$$0 = \frac{dH(x(t), t)}{dt} = \sum_{i=1}^{N} \frac{\partial H}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial H}{\partial t}$$

To compute the paths: use ODE methods to predict and Newton's method to correct.

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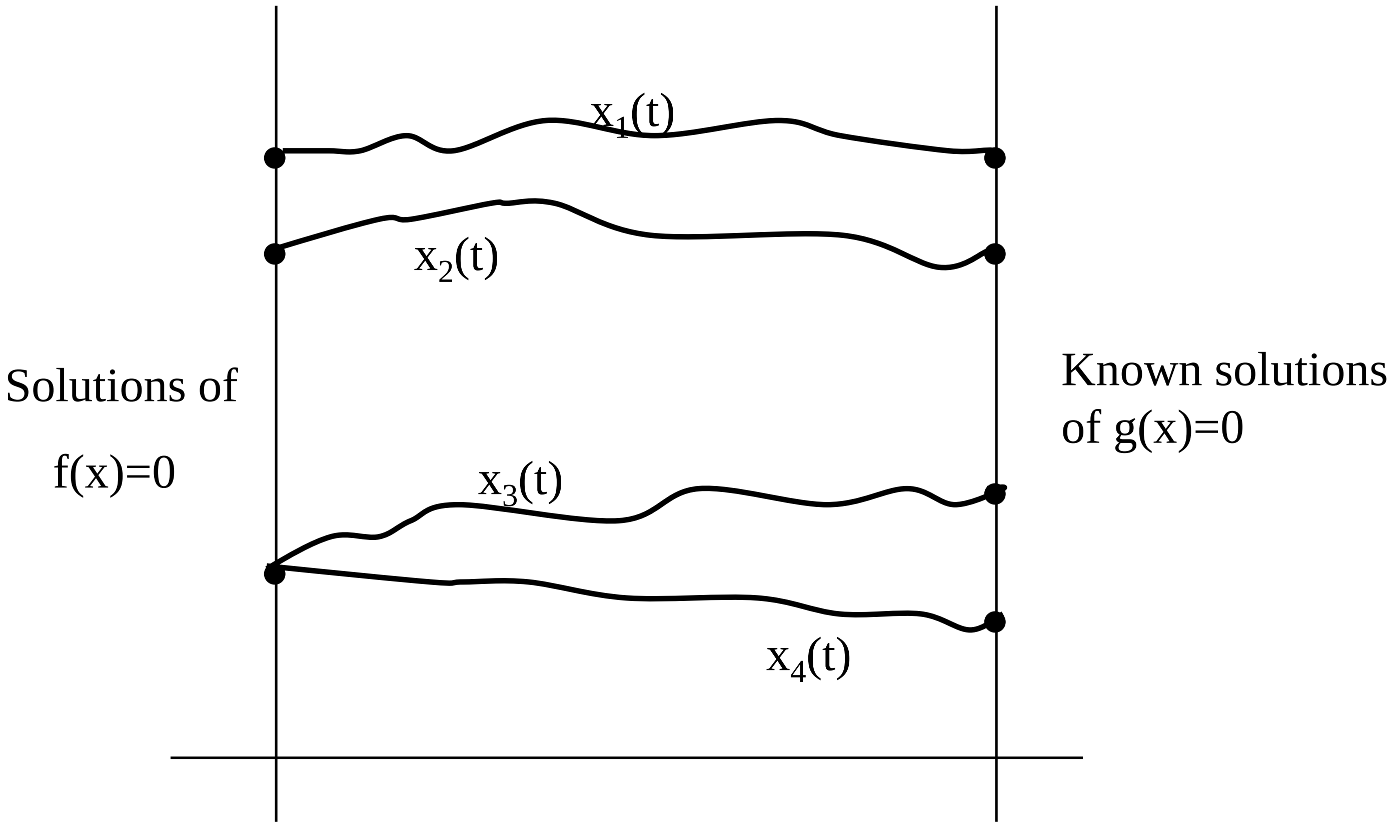
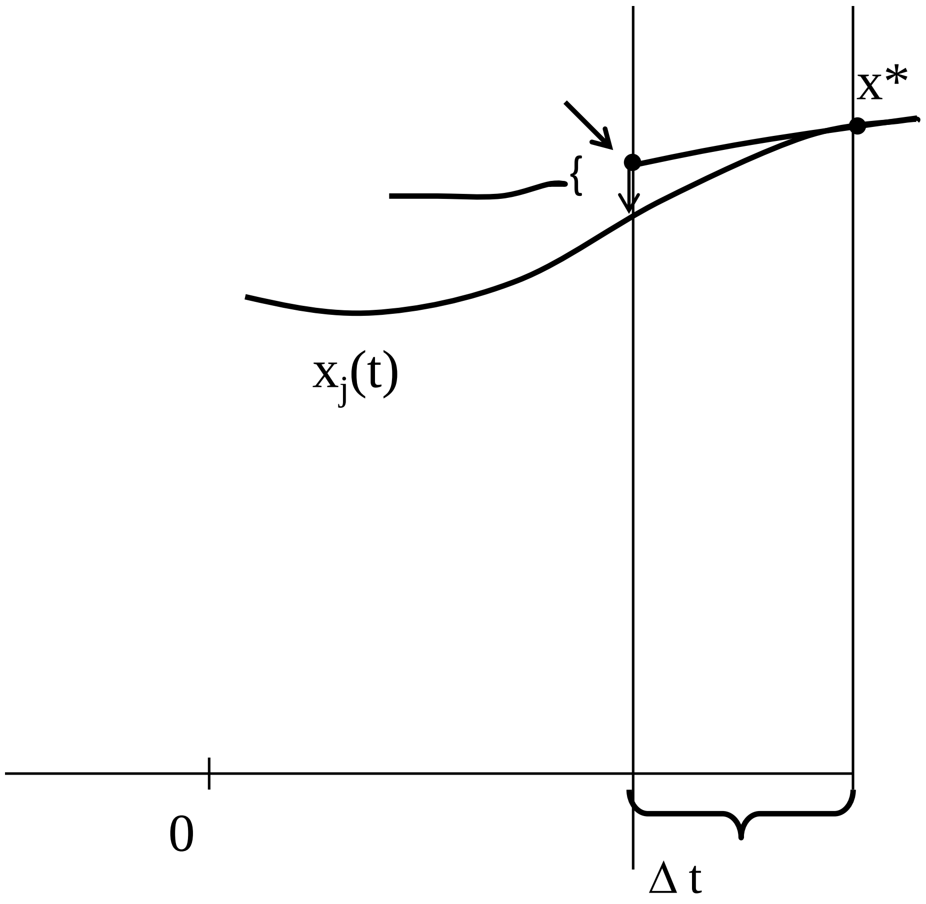


x1(t)
x2(t)
Solutions of
f(x)=0
x3(t)
x4(t)
Known solutions
of g(x)=0



x*
{
$x_j(t)$
0
Δ t

Some Remarks

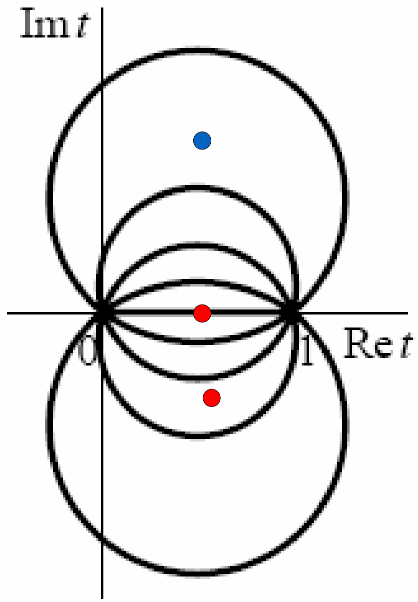
- A Market Inefficiency in the mid 80's
- Numerical versus symbolic methods
  - Numerical methods are inherently uncertain
  - The cost of certainty is not having answers

Uses of algebraic geometry

Simple but extremely useful consequence of algebraicity [A. Morgan (GM R. & D.) and S.]

- Instead of the homotopy $H(x,t) = (1-t)f(x) + tg(x)$

use $H(x,t) = (1-t)f(x) + \gamma t g(x)$



Genericity

- Morgan + S. : if the parameter space is irreducible, solving the system at a random points simplifies subsequent solves: in practice speedups by factors of 100.
  - A. Morgan and A.J. Sommese, Coefficient-parameter polynomial continuation, Appl. Math. Comput. 29 (1989), 123-160.

- A related approach
  - T.Y. Li, T. Sauer, Tim, and J.A. Yorke, The cheater's homotopy: an efficient procedure for solving systems of polynomial equations, SIAM J. Numer. Anal. 26 (1989), 1241-1251.

First Major Use of the Methodology

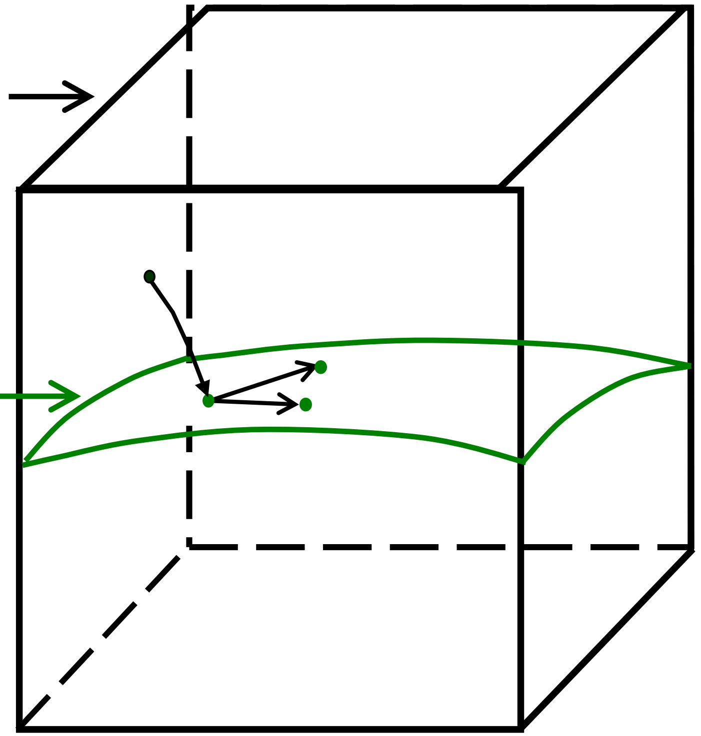
- Kinematics Problem Posed in 1923 by Alt and solved in 1992.
  - C.W. Wampler, A. Morgan, and A.J. Sommese, Complete solution of the nine-point path synthesis problem for four-bar linkages, ASME Journal of Mechanical Design 114 (1992), 153-159.

Timings + Cleverness

Solve by Continuation

All 2-homog. systems

All 9-point systems



"numerical reduction" to test case (done 1 time)

synthesis program (many times)

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Alt's System

$$\left[(\hat{a}-\bar{\delta}_j)x\right]\gamma_j+\left[(a-\delta_j)\hat{x}\right]\hat{\gamma}_j+\delta_j(\hat{a}-\hat{x})+\bar{\delta}_j(a-x)-\delta_j\bar{\delta}_j=0$$

$$\left[(\hat{b}-\bar{\delta}_j)y\right]\gamma_j+\left[(b-\delta_j)\hat{y}\right]\hat{\gamma}_j+\delta_j(\hat{b}-\hat{y})+\bar{\delta}_j(b-y)-\delta_j\bar{\delta}_j=0$$

$$\gamma_j+\hat{\gamma}_j+\gamma_j\hat{\gamma}_j=0$$

in the 24 variables $a, \mathrm{b}, \mathrm{x}, \mathrm{y}, \hat{\mathrm{a}}, \hat{\mathrm{b}}, \hat{\mathrm{x}}, \hat{\mathrm{y}}$ and $\gamma_j, \hat{\gamma}_j$
with j from 1 to 8.

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- 8 degree 2 and 16 degree 3 equations giving 11,019,960,801 paths to follow.

- Freudenstein and Roth (early 50's): use Cramers rule and substitution on the γ variables, we have a system consisting of 8 equations of degree 7. In 1991, this was impractical to solve: $7^8 = 5,764,801$solutions.

Nine-point Problem

Summary

- **Analytical Reduction**
  - Initial formulation .... $\approx 10^{10}$
  - Roth & Freudenstein 5,764,801
  - Our elimination 1,048,576
  - Multi-homogenization 286,720
  - Symmetry 143,360
- **Numerical Reduction**
  - Nondegenerate 4326
  - Roberts cognates.................. 1442
- **Synthesis program tracks 1442 solution paths.**

A point to consider

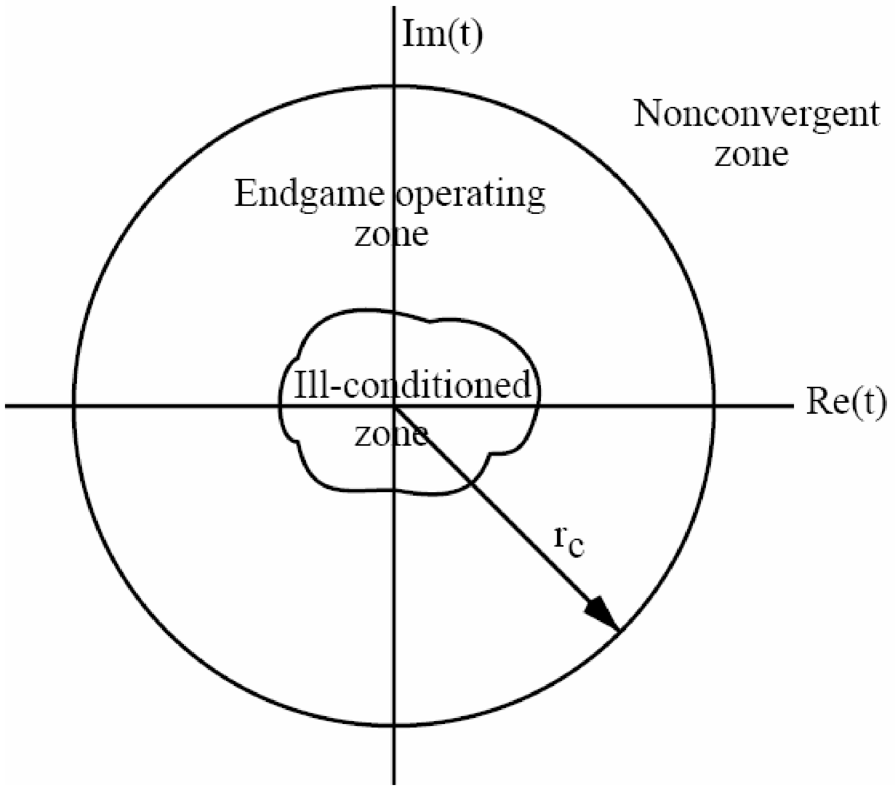
- Not all limits of paths are equal! Singular paths can be much more expensive and difficult to compute.

Difficult means: impossible? Impractical? 10x more expensive?, 100x more expensive?

Endgames (Morgan, S., and Wampler)

- Example: $(x - 1)^2 - t = 0$

We can uniformize around a solution at $t = 0$. Letting $t = s^2$, knowing the solution at $t = 0.01$, we can track around $|s| = 0.1$ and use Cauchy's Integral Theorem to compute x at $s = 0$.



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Continuation's Core Computation - THEN

- Given a system f(x) = 0 of n polynomials in n unknowns, continuation computes a finite set S of solutions such that:
  - any isolated root of f(x) = 0 is contained in S;
  - any isolated root "occurs" a number of times equal to its multiplicity as a solution of f(x) = 0;
  - S is often larger than the set of isolated solutions.

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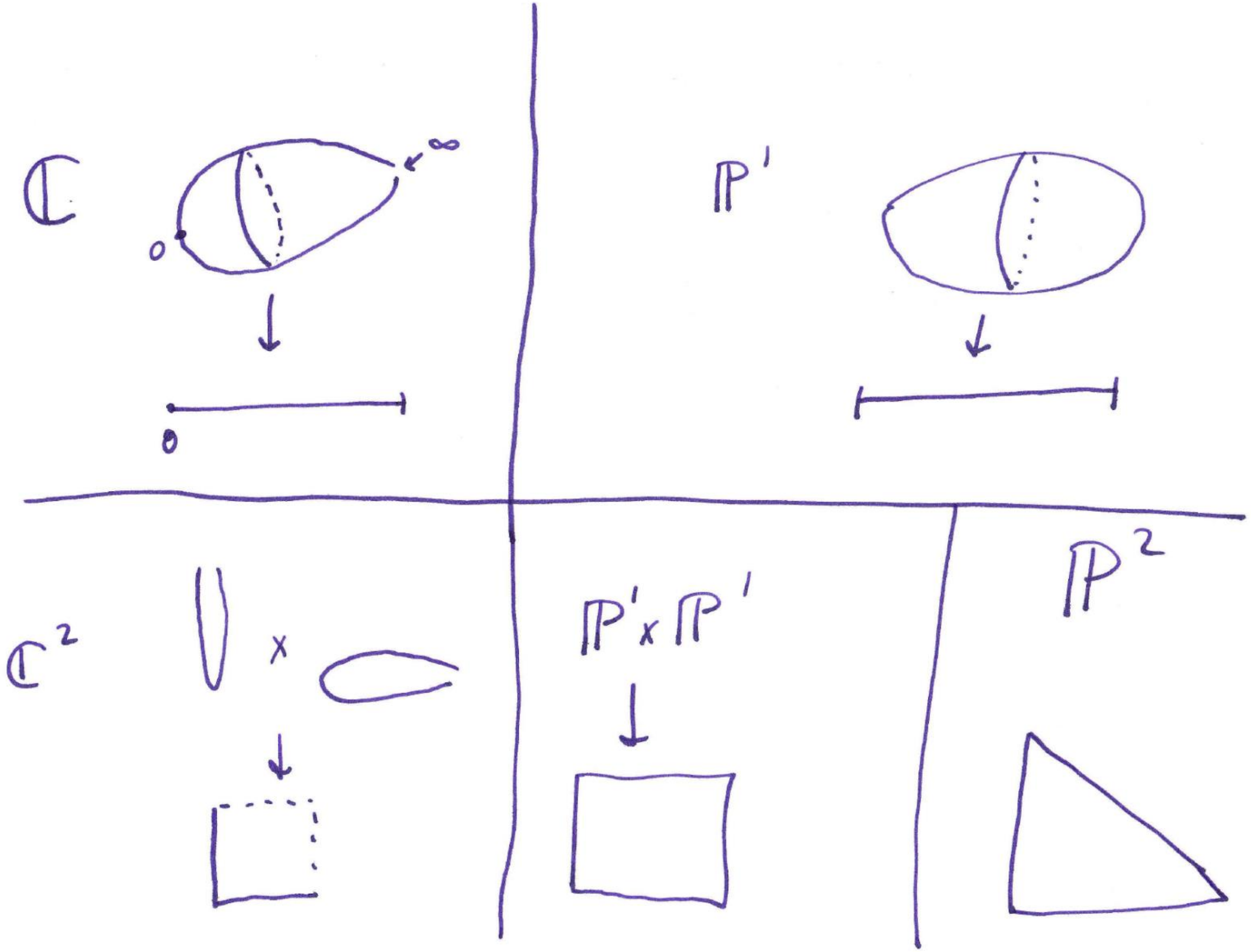
A Guiding Principle

Use Special Homotopies to take advantage of sparseness (*not all endpoints are created equal*).

- Algorithms must be structured – when possible – to avoid paths leading to singular solutions: find a way to never follow the paths in the first place.

Moment Map



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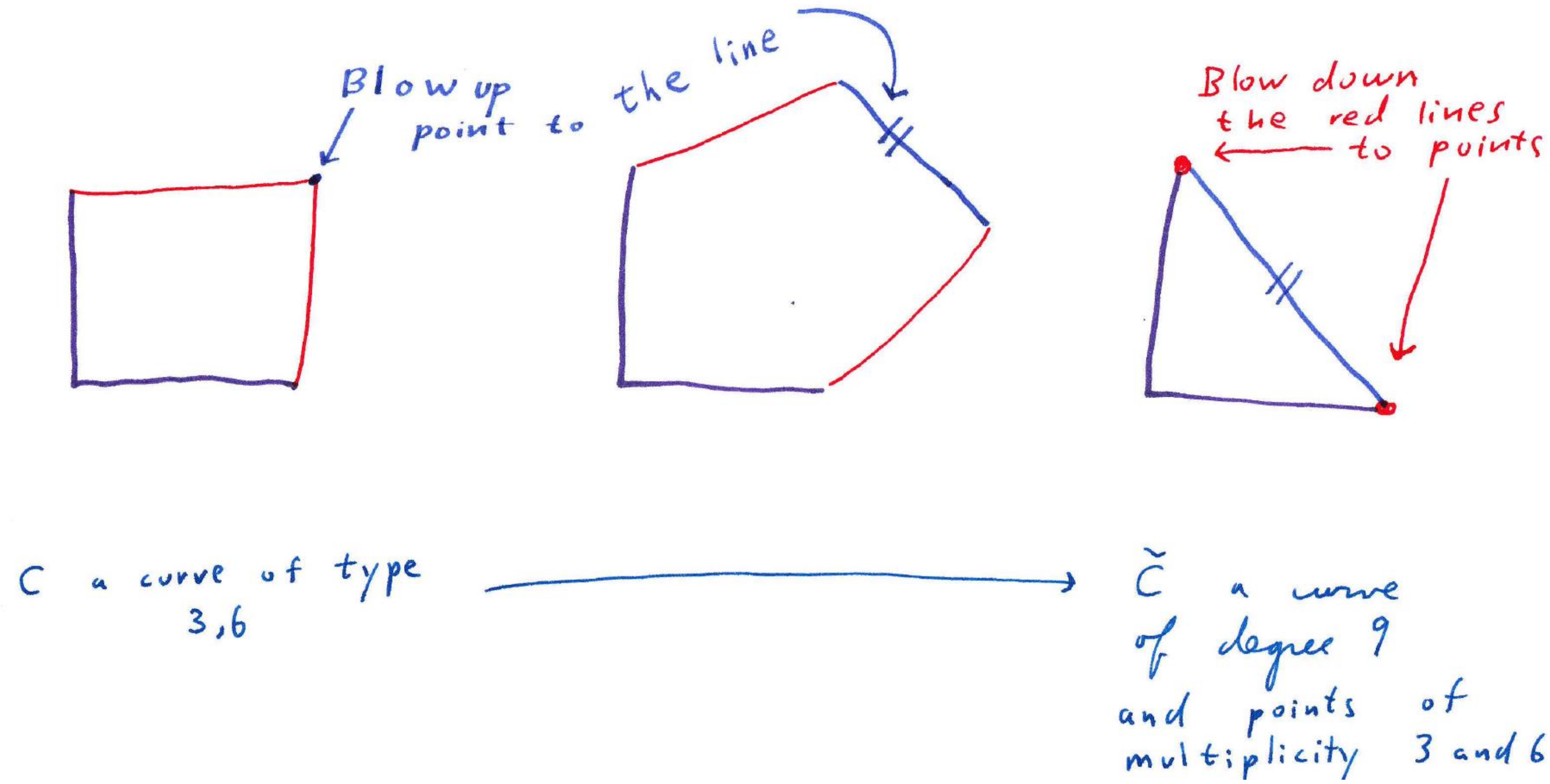
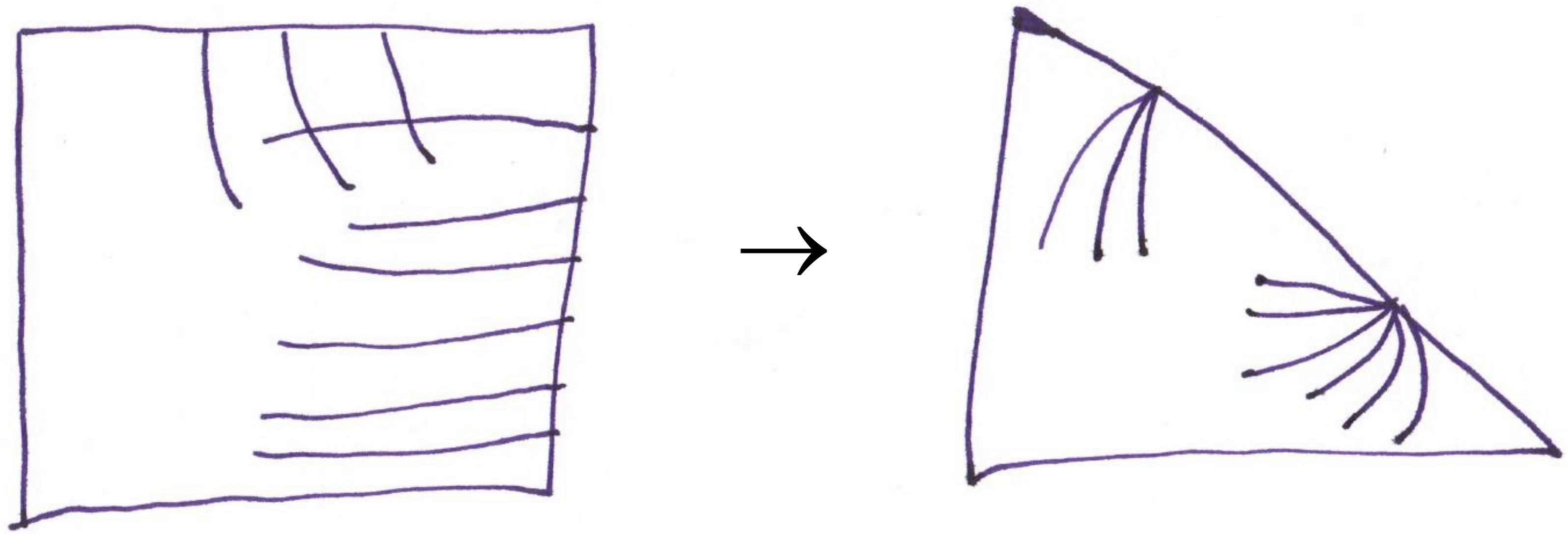
Sparse system treated as nonsparse



Often high multiplicity solutions at infinity



Using Special Structure to avoid bad paths

Multihomogeneity (cites)

- A. Morgan and A.J. Sommese, A homotopy for solving general polynomial systems that respects m-homogeneous structures, Applied Math. and Comput. 24 (1987), 101-113.

Staying on the parameter space in question

- A. Morgan and A.J. Sommese, Coefficient-parameter polynomial continuation, Applied Math. Comput. 29 (1989), 123-160.

Linear Product Structure

- J. Verschelde, and R. Cools, Symbolic homotopy construction, Applied Algebra in Engineering Communication and Computing, 4 (1993), 169-183.

Product Decomposition

- A. Morgan, A.J. Sommese, and C.W. Wampler, A product-decomposition bound for Bezout numbers, SIAM Journal on Numerical Analysis 32 (1995), 1308-1325.

Polyhedral Structure

- T.Y. Li, Numerical solution of polynomial systems by homotopy continuation methods, in *Handbook of Numerical Analysis*, Volume XI, 209-304, North-Holland, 2003.

The current best approach for large systems: Equation-by-Equation Methods

- A.J. Sommese, J. Verschelde, and C.W. Wampler, Solving polynomial systems equation by equation, In *Algorithms in Algebraic Geometry*, edited by A. Dickenstein, F.-O. Schreyer, and A.J. Sommese, vol. 146 of *IMA Volumes in Mathematics and Its Applications*, 133-152, 2007, Springer Verlag.

- J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, Regeneration homotopies for solving systems of polynomials, Mathematics of Computation, 80 (2011) 345-377.

Predator-Prey System

	PHC	HOM4PS-2.0	Bertini	
n	polyhedral	polyhedral	regeneration	parallel regeneration
1	0.6s	0.1s	0.3s	
2	4m57s	7.3s	15.6s	
3	18d10h18m56s	9m32s	9m43s	
4	-	3d8h28m30s	5h22m15s	7m32s
5	-	-	6d16h27m3s	3h41m24s

n = 5 (40 equations & 40 variables): < 80 min. with 200 cores (25 dual Xeon 5410 nodes)

The Core Numerical Computation - NOW

- Realization slowly grew that path crossing is unacceptable and that the core numerical computation of Numerical Algebraic Geometry is:
  - Given a homotopy $H(x;q) = 0$; a "good" path $q(t)$ in the q-variables defined on $(0,1]$; and a point x^* satisfying $H(x^*;q(1))=0$, compute the limit as t goes to 0 of the path $(x(t);q(t))$ starting with $(x(1);q(1)) = (x^*;q(1))$ in the $(x;q)$ space and satisfying $H(x(t);q(t)) = 0$.

In a nutshell:

We need to compute the endpoint of a path!

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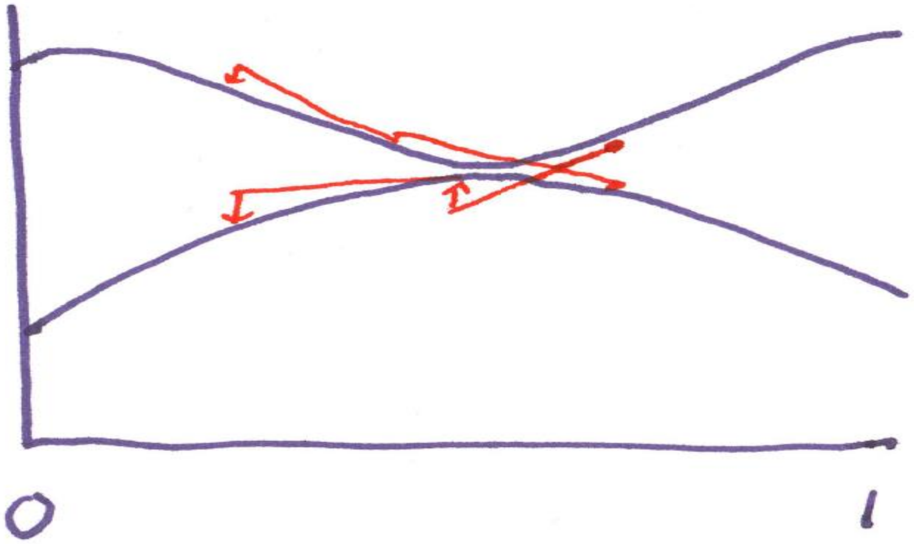
Path-Crossing is dire in modern algorithms!



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Multiprecision

- Not practical in the early 90’s!
  - Highly nontrivial to design and dependent on hardware
  - Hardware too slow

The need for multiprecision

- Why use Multiprecision?
  - Path-crossing is dire in modern algorithms!
  - to ensure that the region where an endgame works is not contained in the region where the numerics break down;
  - to ensure that a polynomial evaluating to zero at a point is the same as the polynomial numerically being approximately zero at the point;
  - To make sure that the consequences of genericity really hold;
  - to prevent the linear algebra in continuation from falling apart.

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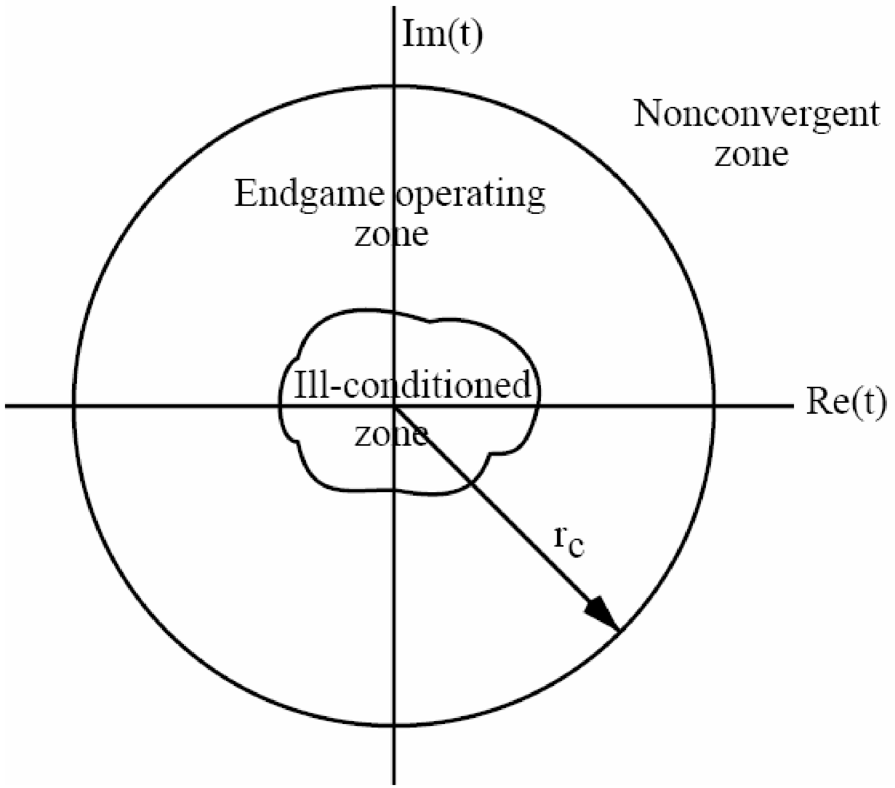
Endgames (Morgan, S., and Wampler)

- Example: $(x - 1)^2 - t = 0$

We can uniformize around a solution at $t = 0$. Letting $t = s^2$, knowing the solution at $t = 0.01$, we can track around $|s| = 0.1$ and use Cauchy's Integral Theorem to compute x at $s = 0$.



Endgames - NOW

- Singular solutions not a side issue, but the main object!

For certain classes of systems of hyperbolic PDEs and solutions with discontinuities, the Cauchy endgame gives an order of magnitude improvement over the standard time-stepping method.

  - W. Hao, J.D. Hauenstein, C.-W. Shu, A.J. Sommese, Z. Xu, Y.-T. Zhang, **A homotopy method based on WENO schemes for solving steady state problems of hyperbolic conservation laws**, J. of Comp. Phys., 250 (2013), 332-346.

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Evaluation

$$p(z) = z^{10} – 28z^9 + 1$$

- To 15 digits of accuracy one of the roots of this polynomial is a = 27.9999999999999. Evaluating p(a) term-by-term to 15 digits, we find that p(a) = -2

  (or, evaluating intelligently p(a) = -0.05784559534077: this uses understanding we do not have in higher dimensions).

- Even with 17 digit accuracy, the approximate root a is a = 27.999999999999905 and we still only have p(a) = -0.01 (or with intelligence: p(a) = -0.0049533155737293).

“Genericity” isn’t possible with only double precision

Near-singular conditions actually arise. For the current best polynomial system to solve Alt’s problem:

- For the nine-point problem, out of 143,360 paths:
  - 1184 paths (0.826%) used higher precision and then dropped back to double precision before starting the endgame
  - 680 paths (0.474%) used at least 96-bit precision and then dropped back to double precision before starting the endgame

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Double precision not enough even at nice solutions!

High condition numbers of 10^7 to 10^9 occur at nice solutions of discretizations of many systems of differential equations, e.g.,

- W. Hao, J.D. Hauenstein, B. Hu,Y. Liu, A.J. Sommese, and Y.-T. Zhang, Continuation along bifurcation branches for a tumor model with a necrotic core, J. Sci. Comp., 53 (2012), 395-413.
- W. Hao, J.D. Hauenstein, B. Hu, T. McCoy,and A.J. Sommese, Computing steady-state solutions for a free boundary problem modeling tumor growth by Stokes equation, J. Comp. and Appl. Math., 237 (2013), 326-334.
- W. Hao, B. Hu, and A.J. Sommese, Cell cycle control and bifurcation for a free boundary problem modeling tissue growth, J. Sci. Comp., 56 (2013), 350-365.

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Double precison versus Higher Precision

Double precision			Multi-precision		
(N_r, N_θ)	$max\ f_i$	order	(N_r, N_θ)	$max\ f_i$	order
(5,32)	6.08e-3	--	(5,32)	6.08e-3	--
(10,64)	1.58e-3	1.94	(10,64)	1.58e-3	1.94
(20,128)	4.00e-4	1.99	(20,128)	4.00e-4	1.99
(40,256)	1.00e-4	2.00	(40,256)	1.00e-4	2.00
(80,512)	2.52e-5	1.99	(80,512)	2.52e-5	1.99
(160,1024)	1.08e-5	1.22	(160,1024)	6.31e-6	2.00
(320,2048)	7.68e-5	-2.83	(320,2048)	1.58e-6	2.01

Bertini

- Bertini is designed to
  - Be efficient and robust, e.g., straightline evaluation, numerics with careful error control
  - With data structures reflecting the underlying geometry
  - Take advantage of parallel hardware
  - To dynamically adjust the precision to achieve a solution with a prespecified error
  - scripting

Using Higher Precision

- One approach is to simply run paths at a higher precision.

This is computationally very expensive!

double (52 bits)	64 bits	96 bits	128 bits	256 bits	512 bits	1024 bits
2.447	32.616	35.456	35.829	50.330	73.009	124.401

From D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, Adaptive multiprecision path tracking, SIAM Journal on Numerical Analysis 46 (2008) 722-746.

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- The theory we use was presented in the article
  - D. Bates, J.D. Hauenstein, S., and W., Multiprecision path tracking, Adaptive multiprecision path tracking, SIAM Journal on Numerical Analysis 46 (2008) 722 - 746.

Positive Dimensional Solution Sets

We now turn to finding the positive dimensional solution sets of a system

$$\begin{bmatrix} \mathrm{f}_1(x_1,\ldots,x_N) \\ \vdots \\ \mathrm{f}_\mathrm{n}(x_1,\ldots,x_N) \end{bmatrix} = 0$$

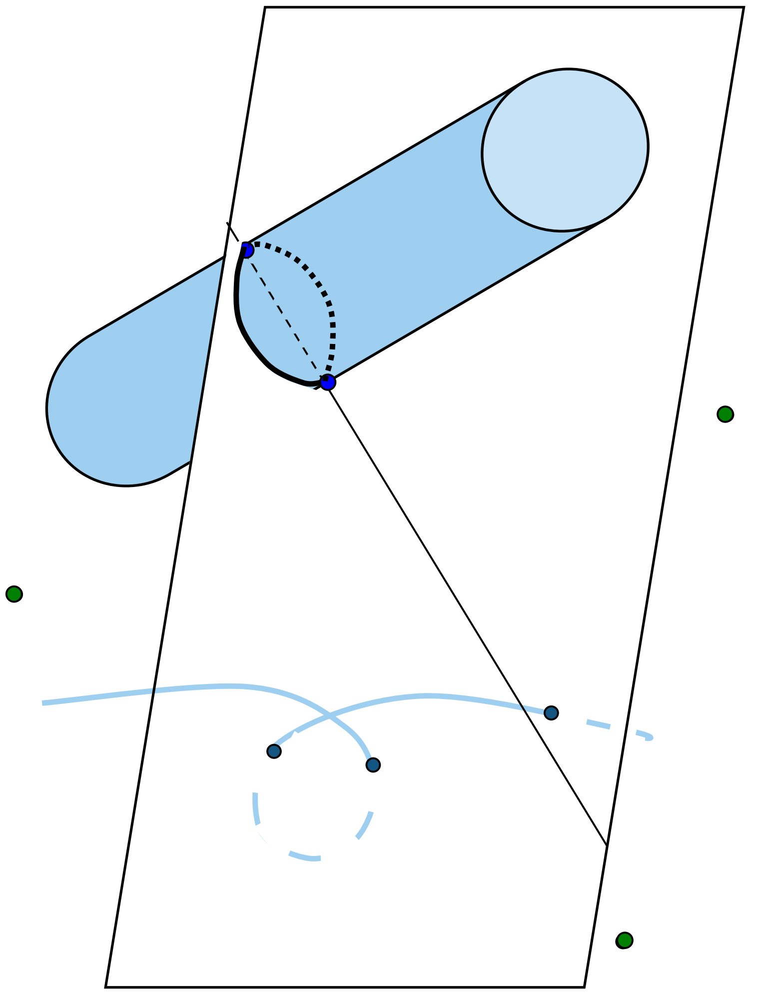
How to represent positive dimensional components?

- S. + Wampler in ’95:
  - Use the intersection of a component with generic linear space of complementary dimension.
  - By using continuation and deforming the linear space, as many points as are desired can be chosen on a component.

In the article “Numerical algebraic geometry, in *The Mathematics of NumericalAnalysis, Park City, Utah, Summer 1995*, ed. by J. Renegar, M. Shub, and S. Smale, Lectures in Applied Math. 32 (1996), 749-763.”

S., Verschelde, and Wampler

- Use a generic flag of affine linear spaces
- to get witness point supersets
- This approach has 19th century roots in algebraic geometry



The Numerical Irreducible Decomposition

Carried out by S., Verschelde, and Wampler in a sequence of articles

- Numerical Irreducible Decomposition
  - SIAM Journal on Numerical Analysis, 38 (2001), 2022-2046.
- An efficient algorithm (with verification) using monodromy
  - SIAM Journal on Numerical Analysis 40 (2002), 2026-2046.
- Intersections of algebraic sets
  - SIAM Journal on Numerical Analysis 42 (2004), 1552-1571.

- A local dimension test letting us deal with a single dimension
  - J.D. Hauenstein, C. Peterson, and A.J. Sommese, SIAM Journal on Numerical Analysis, 47 (2009), 3608-3623.

The Irreducible Decomposition

The solution set $\mathbf{Z} := \mathbf{V}(\mathbf{f})$ decomposes as

$$\bigcup_{\mathrm{i}=0}^{\dim \, Z} \mathbf{Z}_{\mathrm{i}}$$

where $\mathbf{Z}_{\mathrm{i}}$ is pure i-dimensional, and

$$Z_i = \bigcup_{j \in \mathcal{I}_i} Z_{i,j}$$

where

1. each $\mathbf{Z}_{\mathrm{i,j}}$ is irreducible, i.e., for each i, j, the Zariski open and dense set of smooth points of $\mathbf{Z}_{\mathrm{i,j}}$ is connected;

2. $\mathcal{I}_{\mathrm{j}}$ is finite, and no $\mathbf{Z}_{\mathrm{i,j}}$ is contained in the union of all the remaining $\mathbf{Z}_{\mathrm{i',j'}}$.

Witness Point Sets

Witness Point Sets

Given: Polynomial system of n equations, N variables

$$f(x) = \{f_1(x), \ldots, f_n(x)\} = 0, \qquad x = (x_1, \ldots, x_N) \in \mathbb{C}^N$$

Suppose $V \in \mathbb{C}^N$ is

- **a component of the solution set of $f(x) = 0$**
- **irreducible (its regular points are connected in $\mathbb{C}^N$)**
- **degree d and dimension i**

Then
A witness point set for V is

- **d distinct points**
- **that lie on $V \cap L_{N-i}$**
- **where L_{N-i} is a generic linear set of dimension $N - i$.**

In the *numerical irreducible decomposition* we want for each component $Z_{i,j}$

- a finite set $\mathcal{Z}_{i,j}$ of $\deg Z_{i,j}$ points $Z_{i,j} \cap L_{N-i}$, where L_{N-i} is a generic affine linear space $\mathbb{C}^{N-i}$; and
- a "probability one" test for a point $x \in \mathbb{C}^N$ to lie on a given $Z_{i,j}$.

Deflation

The basic idea introduced by Ojika in 1983 is to differentiate the multiplicity away.

- A. Leykin, J. Verschelde, and A. Zhao (Newton's method with deflation for isolated singularities of polynomial systems, Theoret. Comput. Sci., 359 (2006),111–122) gave an algorithm for an isolated point that they showed terminated.

Given a system f, replace it with

$$\begin{bmatrix} f(x) \\ Jf(x)\cdot z \\ A\cdot z+b \end{bmatrix} = 0$$

Repeat as necessary.

This led to much work and improvement by many mathematicians. One very important development in understanding this and how to compute multiplicities is

- B.H. Dayton and Z. Zeng, Computing the multiplicity structure in solving polynomial systems, ISSAC'05, 116-123, ACM, New York, 2005.

See also

- B.H. Dayton, T.-Y. Li, and Z. Zeng, Multiple zeros of nonlinear systems, Math. Comp. 80 (2011), 2143-2168.

The best current approach for dealing with nonreduced components is given by

- J.D. Hauenstein and C.W. Wampler, Isosingular sets and deflation, Found. of Comput. Math., 13 (2013), 371-403.

A related article

- J.D. Hauenstein and C.W. Wampler, Numerically intersecting algebraic varieties via witness sets, Appl. Math. and Comp., (2013), 5730-5742.

is used to complete the process of computing the numerical irreducible decomposition of intersections by producing the witness systems for the witness sets.

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Real algebraic sets

There is an extensive literature on algebraic methods:

- S. Basu, R. Pollack, and M.-F. Roy, Algorithms in real algebraic geometry, Second edition, *Algorithms and Computation in Mathematics*, 10. Springer-Verlag, Berlin, 2006.

Numerical approach

Curves:

- Y. Lu, D.J. Bates, A.J. Sommese, and C.W. Wampler, Finding all real points of a complex curve, In *Algebra, Geometry and Their Interactions*, edited by A. Corso, J. Migliore, and C. Polini, Cont. Math. 448 (2007), 183-205, American Mathematics Society.

Real surfaces:

- G.M. Besana, S. Di Rocco, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, Cell decomposition of almost smooth real algebraic surfaces, Numerical Algorithms, 63(2013), 645-678.

Implemented in Bertini_real:

- D.A. Brake, D.J. Bates, W. Hao, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, Bertini_real: Numerical decomposition of real algebraic curves and surfaces, ACM Trans. on Math. Software, to appear.

Related:

- The CGAL Project. *CGAL User and Reference Manual*. CGAL Editorial Board, 4.10 edition, 2017.
- Work of Bernhard Mourrain and his group

Algebraic Geometry Computations

- D.J. Bates, W. Decker, J.D. Hauenstein, C. Peterson, G. Pfister, F.-O. Schreyer, A.J. Sommese, and C.W. Wampler, Probabilistic algorithms to analyze the components of an affine algebraic variety, Applied Mathematics and Computation, 231 (2014), 619-633.

- Many papers by Bates, Brake, Dayton, Hauenstein, Leykin, Li, Oeding, Ottaviani, Peterson, Rodriguez, Sommese, Sottile, Sturmfels, Verschelde, Wampler, Zeng, . . .

Questions?

Happy Birthday Charles!