Kinematic mapping - recent results and applications

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POLYNOMIALS KINEMATICS AND ROBOTICS

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Overview

Euclidean displacement:

$$\gamma\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{a} \tag{1}$$

A proper orthogonal 3×3 matrix, $\mathbf{a} \in \mathbb{R}^3 \ldots$ vector

group of Euclidean displacements: $\mathrm{SE}(3)$

$$\begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} \mapsto \begin{bmatrix} 1 & \mathbf{o}^T \\ \mathbf{a} & \mathbf{A} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}. \tag{2}$$

Study's kinematic mapping $\varkappa$:

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pre-image of **x** is the displacement α

$$\frac{1}{\Delta}\begin{bmatrix} \Delta & 0 & 0 & 0 \\ p & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ q & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ r & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \tag{3}$$

$$\begin{aligned} p &= 2(-x_0y_1+x_1y_0-x_2y_3+x_3y_2), \\ q &= 2(-x_0y_2+x_1y_3+x_2y_0-x_3y_1), \\ r &= 2(-x_0y_3-x_1y_2+x_2y_1+x_3y_0), \end{aligned} \tag{4}$$

$\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2.$

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$$S_6^2: \quad x_0y_0+x_1y_1+x_2y_2+x_3y_3=0, \qquad x_i \quad \text{not all} \quad 0 \tag{5}$$

$[x_0 : \cdots : y_3]^T$ *Study parameters* = parametrization of $SE(3)$ with dual quaternions

How do we get the Study parameters when a proper orthogonal matrix $\mathbf{A} = [a_{ij}]$ and the translation vector $\mathbf{a} = [a_k]^T$ are given?

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Rotation part:

$$\begin{aligned} x_0 : x_1 : x_2 : x_3 &= 1 + a_{11} + a_{22} + a_{33} : a_{32} - a_{23} : a_{13} - a_{31} : a_{21} - a_{12} \\ &= a_{32} - a_{23} : 1 + a_{11} - a_{22} - a_{33} : a_{12} + a_{21} : a_{31} + a_{13} \\ &= a_{13} - a_{31} : a_{12} + a_{21} : 1 - a_{11} + a_{22} - a_{33} : a_{23} + a_{32} \\ &= a_{21} - a_{12} : a_{31} + a_{13} : a_{23} - a_{32} : 1 - a_{11} - a_{22} + a_{33} \end{aligned} \tag{6}$$

Translation part:

$$\begin{aligned} 2y_0 &= a_1 x_1 + a_2 x_2 + a_3 x_3, & 2y_1 &= -a_1 x_0 + a_3 x_2 - a_2 x_3, \\ 2y_2 &= -a_2 x_0 - a_3 x_1 + a_1 x_3, & 2y_3 &= -a_3 x_0 + a_2 x_1 - a_1 x_2. \end{aligned} \tag{7}$$

Invariant geometric objects in $\mathbb{P}^7$

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Study quadric S_6^2

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pencil of quadrics $\mathcal{D}$

$$\mathcal{D} = \lambda S_6^2 + \mu\mathcal{N}$$

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more properties: J. Selig, Geometric Fundamentals of Robotics, 2nd. ed. Springer 2005

Dual Quaternions

generated by units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\varepsilon\mathbf{i}$, $\varepsilon\mathbf{j}$, $\varepsilon\mathbf{k}$ $\in \mathbb{R}$:

$$Q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 + \varepsilon y_0 + \varepsilon\mathbf{i}y_1 + \varepsilon\mathbf{j}y_2 + \varepsilon\mathbf{k}y_3$$

$\varepsilon \dots$ dual unit $\quad \varepsilon^2 = 0$

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conjugate dual quaternion

$$\overline{Q} = x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3 + \varepsilon y_0 - \varepsilon\mathbf{i}y_1 - \varepsilon\mathbf{j}y_2 - \varepsilon\mathbf{k}y_3$$

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McCarthy, ...

Image space transformations

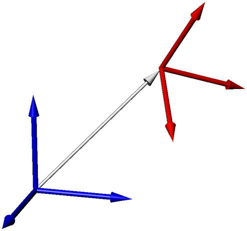
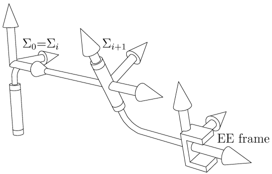


Abbildung: Fixed and moving coordinate systems



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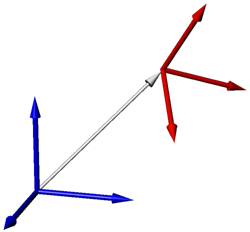

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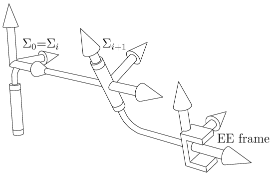

Abbildung: Robot coordinate systems

- The relative displacement α depends on the choice of fixed and moving frame.
- Coordinate systems are usually attached to the base and the end-effector of a mechanism.
- Changes of fixed and moving frame induce transformations on S_6^2, impose a geometric structure on S_6^2.
- Canonical frames.

Image space transformations

$$\mathbf{y} = \mathbf{T}_f \mathbf{T}_m \mathbf{x}, \quad \mathbf{T}_m = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}, \quad \mathbf{T}_f = \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{D} & \mathbf{C} \end{bmatrix}, \tag{8}$$

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$$\mathbf{A} = \begin{bmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ m_1 & m_0 & m_3 & -m_2 \\ m_2 & -m_3 & m_0 & m_1 \\ m_3 & m_2 & -m_1 & m_0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} m_4 & -m_5 & -m_6 & -m_7 \\ m_5 & m_4 & m_7 & -m_6 \\ m_6 & -m_7 & m_4 & m_5 \\ m_7 & m_6 & -m_5 & m_4 \end{bmatrix} \tag{9}$$

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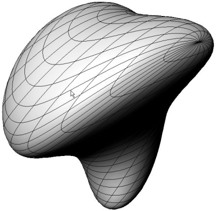
$$\mathbf{C} = \begin{bmatrix} f_0 & -f_1 & -f_2 & -f_3 \\ f_1 & f_0 & -f_3 & f_2 \\ f_2 & f_3 & f_0 & -f_1 \\ f_3 & -f_2 & f_1 & f_0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} f_4 & -f_5 & -f_6 & -f_7 \\ f_5 & f_4 & -f_7 & f_6 \\ f_6 & f_7 & f_4 & -f_5 \\ f_7 & -f_6 & f_5 & f_4 \end{bmatrix} \tag{10}$$

and **O** is the four by four zero matrix.

- $\mathbf{T}_m$ and $\mathbf{T}_f$ commute
- $\mathbf{T}_m$ and $\mathbf{T}_f$ induce transformations of P^7 that fix S_6^2, the exceptional generator $\mathcal{E}$ the exceptional quadric $\mathcal{Y}$ the Null-cone $\mathcal{N}$ and the pencil $\mathcal{D} = \lambda S_6^2 + \mu \mathcal{N}$
- Clifford translations on S_6^2

Constraint varieties

$\mapsto$



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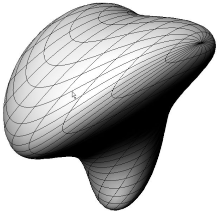

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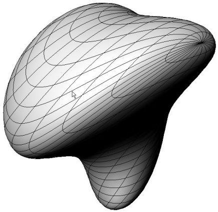

- a constraint that removes one degree of freedom maps to a hyper-surface in $\mathbb{P}^7$
- a set of constraints corresponds to a set of polynomial equations

Global Kinematics - Methods: Derivation of constraint equations

Three methods:

- Geometric constraint equations
- Elimination method
- Linear implicitization algorithm

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1. Constraint equations are algebraic equations as long as no helical joint is in the mechanism
2. Derive at first the constraint equations for a canonical chain (= best adapted coordinate systems to base and end effector)
3. Change of frames is linear in algebraic (dual quaternion) parameters

1. Geometric constraint equations

For simple chains

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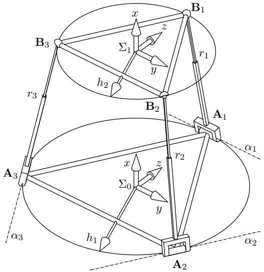

Abbildung: 3-RPS parallel robot

each leg has two constraints:

1. plane constraint
2. distance constraint

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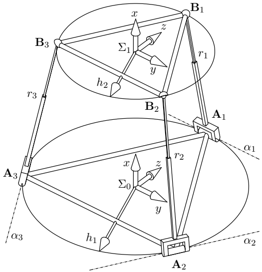

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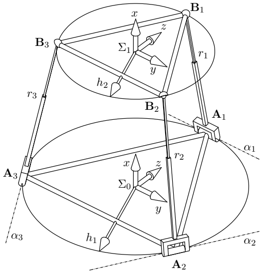

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This method was used 20 years ago to derive the constraint equations of the Stewart Gough platform and solve the DK

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Example:

3-R chain $\rightarrow$ 3 constraint equations describing a 3-dim geometric object sitting on the Study quadric (incomplete!!)

3. Linear implicitization algorithm

D. R. Walter and M. L. H. On Implicitization of Kinematic Constraint Equations. Machine Design Research, 26:218-226,2010

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Basic idea:

- If one has an implicit representation of a geometric object and a parametric expression, then the parametric expression must fulfill the implicit equation.
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- Substitution of the parametric equation into a general polynomial of a degree n yields an (overdetermined) set of linear equations in the coefficients of the implicit equation.

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Example:

The complete description of a 3-R chain needs 9 equations.

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- Identification of different operation modes
- New form of polynomial motion interpolation

Global Singularities

Let $V \in k^n$ be a constraint variety and let $p = [p_0, \ldots, p_7]^T$ be a point on V. The *tangent space* of V at p, denoted $T_p(V)$, is the variety

$$T_P(V) = \mathbf{V}(d_p(f)) \colon f \subset \mathbf{I(V)} \tag{11}$$

of linear forms of all polynomials contained in the ideal $\mathbf{I(V)}$ in point p.

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The local degree of freedom is defined as $\dim T_p(V)$.

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$$T_P(V) = \mathbf{V}(d_p(f))\colon f \subset \mathbf{I}(\mathbf{V}) \tag{11}$$

of linear forms of all polynomials contained in the ideal $\mathbf{I}(\mathbf{V})$ in point p.

The local degree of freedom is defined as dim $T_p(V)$.

Jacobian of the set of constraint equations:

$$\mathbf{J}(f_j) = \left(\frac{\partial f_j}{x_i}, \frac{\partial f_j}{y_i} \right), \tag{12}$$

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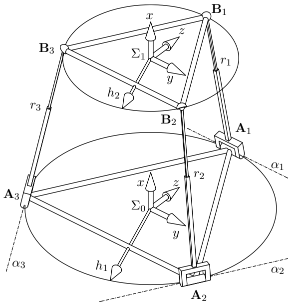
the manipulator is in a singular pose:

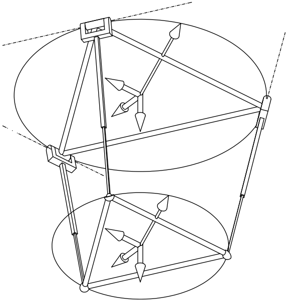
$$S : \det \mathbf{J} = 0$$

yields the global singularity variety

Constraint equations of inverted kinematic chains

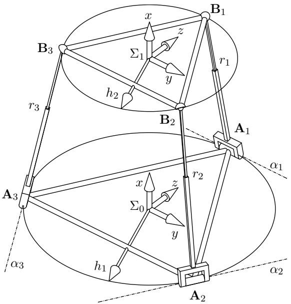


$\rightarrow\rightarrow$

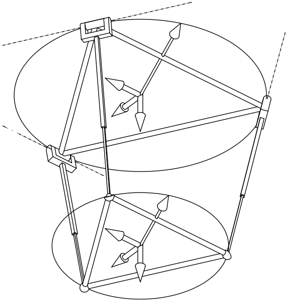


- What happens to the constraint equations when the manipulator is upside down??
- Change of platform and base!!

Constraint equations of inverted kinematic chains



$\longrightarrow$



- What happens to the constraint equations when the manipulator is upside down??
- Change of platform and base!!
- Quaternion conjugation!!

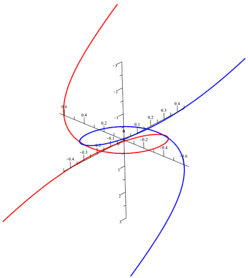
Conjugation - invariant objects

Line v and 5-dim. Subspace w in $\mathbb{P}^7$

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s, \quad w = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} t_2 + \ldots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t_6,$$

with $t, s, t_1, t_2, \ldots, t_6 \in \mathbb{R}$

- Inverting a constraint $\rightarrow$ projective transformation in the image space
- topology of the objects is invariant
- rulings of $\mathcal{Y}$ are interchanged $\rightarrow$ “chirality” in kinematics
- geometric constraints dualize



Cardan Motion (Trammel-, Elliptic- motion) $\leftrightarrow$ Oldham motion

Operation Modes - Ideal Decomposition

Example: 3-UPU-Parallel Manipulator

Abbildung: 3-UPU-Model

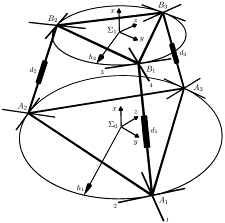


Abbildung: 3-UPU-manipulator

Six constraint equations

1. 3 (quadratic) sphere constraint equations $g_1 - g_3$
2. 3 (bilinear) plane constraint equations $g_4 - g_6$

$$g_4 : \; 4\,x_1\,y_1 + x_2\,y_2 + \sqrt{3}\,x_2\,y_3 + \sqrt{3}\,x_3\,y_2 + 3\,x_3\,y_3 = 0 \tag{13}$$

$$g_5 : \; 4\,x_1\,y_1 + x_2\,y_2 - \sqrt{3}\,x_2\,y_3 - \sqrt{3}\,x_3\,y_2 + 3\,x_3\,y_3 = 0 \tag{14}$$

$$g_6 : \; x_1\,y_1 + x_2\,y_2 = 0 \tag{15}$$

the subsystem $\mathcal{J} =< g_4, g_5, g_6, g_7 >$ is independent of the design parameters splits into 10 subsystems

$$\mathcal{J}_1 = \langle y_0, y_1, y_2, y_3 \rangle, \ \mathcal{J}_2 = \langle x_0, y_1, y_2, y_3 \rangle, \ \mathcal{J}_3 = \langle y_0, x_1, y_2, y_3 \rangle, \ \mathcal{J}_4 = \langle x_0, x_1, y_2, y_3 \rangle,$$

$$\mathcal{J}_5 = \langle y_0, y_1, x_2, x_3 \rangle, \ \mathcal{J}_6 = \langle x_0, y_1, x_2, x_3 \rangle, \ \mathcal{J}_7 = \langle y_0, x_1, x_2, x_3 \rangle,$$

$$\mathcal{J}_8 = \langle x_2 - i\, x_3, y_2 + i\, y_3, x_0\, y_0 + x_3\, y_3, x_1\, y_1 + x_3\, y_3 \rangle,$$

$$\mathcal{J}_9 = \langle x_2 + i\, x_3, y_2 - i\, y_3, x_0\, y_0 + x_3\, y_3, x_1\, y_1 + x_3\, y_3 \rangle,$$

$$\mathcal{J}_{10} = \langle x_0, x_1, x_2, x_3 \rangle.$$

- Manipulator with the same actuator lengths has 72 solutions of the direct kinematics.
- Manipulator with different actuator lengths has 78 solutions of the direct kinematics.

Relations between the different components, which relate to different operation modes of the manipulator

	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$	$\mathcal{K}_4$	$\mathcal{K}_5$	$\mathcal{K}_6$	$\mathcal{K}_7$
$\mathcal{K}_1$	3	2	2	1	1	0	0
$\mathcal{K}_2$	2	3	1	2	0	1	-1
$\mathcal{K}_3$	2	1	3	2	0	-1	1
$\mathcal{K}_4$	1	2	2	3	-1	-1	-1
$\mathcal{K}_5$	1	0	0	-1	3	2	2
$\mathcal{K}_6$	0	1	-1	-1	2	3	-1
$\mathcal{K}_7$	0	-1	1	-1	2	-1	3

Path planning in kinematic image space

$$\mathbf{x}' = \mathbf{M}\mathbf{x}$$

$$\mathbf{d} = [x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3]^T$$

d point in seven dimensional projective space P^7 fulfills the quadratic *Study condition*

$$x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0, \tag{16}$$

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$$\mathbf{M} := \kappa^{-1}(\mathbf{d}) = \frac{1}{\Delta}\begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & x_0^2 + x_1^2 - x_3^2 - x_2^2 & -2x_0x_3 + 2x_2x_1 & 2x_3x_1 + 2x_0x_2 \\ t_2 & 2x_2x_1 + 2x_0x_3 & x_0^2 + x_2^2 - x_1^2 - x_3^2 & -2x_0x_1 + 2x_3x_2 \\ t_3 & -2x_0x_2 + 2x_3x_1 & 2x_3x_2 + 2x_0x_1 & x_0^2 + x_3^2 - x_2^2 - x_1^2 \end{bmatrix} \tag{17}$$

where $\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2$ and

$$\begin{aligned} t_1 &= 2x_0y_1 - 2y_0x_1 - 2y_2x_3 + 2y_3x_2, \\ t_2 &= 2x_0y_2 - 2y_0x_2 - 2y_3x_1 + 2y_1x_3, \\ t_3 &= 2x_0y_3 - 2y_0x_3 - 2y_1x_2 + 2y_2x_1. \end{aligned} \tag{18}$$

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exceptional three space

$$x_0 = x_1 = x_2 = x_3 = 0$$

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"*extended kinematic mapping*"

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$$\mathbf{M}(\mathbf{a}) = \mathbf{M}(\mathbf{b}) \tag{19}$$

$$\{\mathbf{a} + \lambda(0, 0, 0, 0, a_0, a_1, a_2, a_3) \mid \lambda \in \mathbb{R}\}.$$

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Theorem

The fiber of point $\mathbf{a} = [a_0, \ldots, a_7] \in P^7 \setminus E$ *with respect to the extended inverse kinematic map* κ^{-1} *is a straight line through* $\mathbf{a}$ *that intersects the exceptional generator* E *in* $[0,0,0,0,a_0,\ldots,a_3]$.

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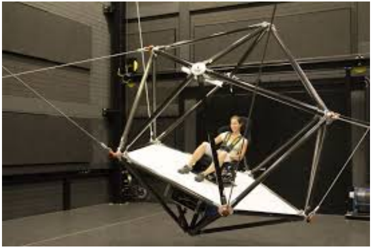
Properties of κ^{-1}

- κ^{-1} is quadratic, the degree of trajectories is at most twice the degree of the interpolant in P^7
- one can achieve a geometric continuity of order n for the motion with trajectories of degree $2(n+1)$
- At possible intersection points of interpolant and exceptional generator E, the map κ^{-1} becomes singular and a degree reduction of the trajectories occurs

Cable driven parallel manipulators

Cable driven parallel manipulators



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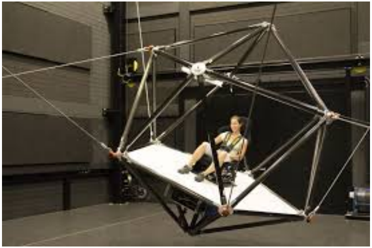

Much more complicated than DK Stewart-Gough platform

DK solutions for cable configuration

Number of cables	2	3	4	5
Number of solutions over $\mathbb{C}$	24	156	216	140

Numbers are due to additional equilibrium constraints!

Thanks for your attention!